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CONCENTRATION/DENSITY SHOCKS IN AN INERTIALLY-COUPLED TWO-PHASE DISPERSION

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Abstract—We analyze a plane shock wave propagating into a homogeneous two-phase mixture in which the gas density is small compared with the liquid density. Inertial effects are assumed to govern the mechanics and "added mass" effects are represented by the methods of Geurst and Wallis. Expressions are obtained for the shock speed, as well as void fraction, pressure and density jumps across small-amplitude shocks. A method is presented for predicting the behavior of large-amplitude shocks; their characteristics and conditions for their existence are discussed.

Key Words: exertia, inertial coupling, finite waves, jump conditions

FORMULATION OF THE PROBLEMS: SHOCK CONDITIONS

We consider an inertially-coupled two-phase dispersion, assuming that the continuous phase is incompressible and taking into account the compressibility of the dispersed phase. We also assume that the density of the dispersed phase is much lower than the density of the continuous phase, i.e.

$$\rho_2 \ll \rho_1, \qquad [1]$$

so that the model considered below corresponds to the dispersion of gas bubbles in an incompressible fluid.

Our purpose is to analyze the propagation of a concentration (voidage) jump associated with a jump in density of the dispersed phase [it has been stressed by Sergeev & Wallis (1991) that concentration shock waves propagating relative to the dispersed phase in an inertially-coupled dispersion are possible only in the case when at least one of the phases is compressible].

The model below can be used to describe shock waves in bubbly liquids. Observations of such shock waves have been reported in a number of publications since the work of Campbell & Pitcher (1958); very good observations and experimental data can be found, for example, in Noordzij (1973) and Noordzij & van Wijngaarden (1974). Nevertheless, no theoretical considerations based on the model of inertially-coupled dispersions have been provided for shock waves, although an inertial interphase interaction prevails in many practical cases.

We consider the case when there is no relative motion in the two-phase media ahead of the shock so that $w^0 = v_1^0 - v_2^0 = 0$ (here v_1 and v_2 are, respectively, the velocities of the continuous and dispersed phase, w is the relative velocity and the superscript "0" herein denotes the hydrodynamic parameters in front of the shock). Choosing the appropriate coordinate system, we can assume that the two-phase medium is motionless in front of the shock.

The shock conditions for the (reversible) concentration/density discontinuity in the case of a barotropically compressible dispersed phase have been derived by Sergeev & Wallis (1991), on the basis of the closed system of macroscopic equations for the potential flow of a two-phase dispersion obtained by Geurst (1985a, b, 1986) and Wallis (1989a, b, 1990).

These equations correspond to conservation of mass, momentum and energy, it being implied that compression is sufficiently rapid that kinetic energy due to relative motion (i.e. representing added mass effects) is directly convertible and is not immediately dissipated. Relative motion also contributes to the net stress tensor in the momentum balance. This is reasonable since mutual

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inertial forces dominate drag forces during rapid transients. There might then be expected to be a "relaxation" region behind the immediate shock, in which the phases tend to come to equilibrium, eventually achieving equal velocities if there are no further causes of relative motion.

With the above assumptions in the coordinate system connected with the shock, the closed set of the one-dimensional shock conditions can be written as follows (absence of the superscript indicates values behind the shock):

$$\varepsilon_1 v_1 = -\varepsilon_1^0 D, \qquad [2]$$

$$\rho_2 \varepsilon_2 v_2 = -\rho_2^0 \varepsilon_2^0 D, \qquad [3]$$

$$\varepsilon_1 v_1^2 + H(\varepsilon_2) w^2 + \frac{1}{\rho_1} p(\rho_2) = \varepsilon_1^0 D^2 + \frac{1}{\rho_1} p(\rho_2^0) \qquad [4]$$

and

$$\frac{1}{2} v_1^2 + E(\varepsilon_2) v_1 w + \frac{1}{2} F(\varepsilon_2) w^2 + \frac{1}{\rho_1} p(\rho_2) = \frac{1}{2} D^2 + \frac{1}{\rho_1} p(\rho_2^0). \qquad [5]$$

Here p is the pressure in the dispersed phase; it is assumed to depend only on the density of the dispersed phase so we consider barotropic behavior of the gas bubbles. ε_1 and ε_2 are the volumetric concentrations of the continuous and dispersed phases, respectively, so that

$$\varepsilon_1 + \varepsilon_2 = 1. \qquad [6]$$

The propagation speed of the shock D relative to the motionless undisturbed two-phase media is to obtained as part of the solution of the problem.

For the considered inertially-coupled dispersion the interphase interaction is described by the terms involving the *exertia* $E(\varepsilon_2)$ defined by Wallis (1989a); the functions H and F introduced in Sergeev & Wallis (1991) are represented through the exertia as follows:

$$H = \frac{1}{2} \varepsilon_1^2 \frac{dE}{d\varepsilon_2} + \varepsilon_1 E, \quad F = \varepsilon_1 \frac{dE}{d\varepsilon_2} - E. \qquad [7]$$

One of the basic assumptions used by Wallis (1989b) to develop the general equations of inertially-coupled dispersions is that the mean potential gradient associated with an average flux of the continuous phase relative to a matrix of particles is β times what it would be without the particles. The *exertia* which appears in the basic equations is then

$$E = \beta \varepsilon_1 - 1.$$

β was derived as an approximate function of the void fraction by Maxwell (1881), who considered the analogous problem of electrical conduction past a matrix of non-conducting spheres. The value of β found by Maxwell leads to E in the form

$$E = \frac{\varepsilon_2}{2}. \qquad [8]$$

Relationship [8] has been confirmed by Turner (1976) from electrical conductivity measurements in a fluidized bed.

Now, functions [7] reduce to

$$H = \frac{1}{4}(1 - \varepsilon_2^2) \qquad [9]$$

and

$$F = \frac{1}{2}(1 - 2\varepsilon_2). \qquad [10]$$

Following the approach usually accepted in gas dynamics (e.g. Courant & Friedrichs 1948), it is most convenient to introduce new variables:

$$\tau = \frac{1}{\varepsilon_1} = \left(\frac{1}{1 - \varepsilon_2}\right), \quad \theta = \frac{\rho_2^0}{\rho_2}. \qquad [11]$$

(in front of the shock, $\tau = \tau^0 = 1/\varepsilon_1^0$ and $\theta = \theta^0 = 1$). The ranges of the variables τ and θ are

$$1 \leqslant \tau \leqslant +\infty, \quad 0 < \theta < +\infty. \tag{12}$$

Taking into account the continuity conditions [2] and [3], the velocities of the phases and the relative velocity can now be written as

$$v_1 = M\tau, \quad v_2 = M\tau\theta\frac{\tau^0 - 1}{\tau - 1} \tag{13}$$

and

$$w = M\tau\left(1 - \theta\frac{\tau^0 - 1}{\tau - 1}\right), \tag{14}$$

where we have introduced the parameter

$$M = \varepsilon_1 v_1 = -\varepsilon_1^0 D = \text{const.} \tag{15}$$

Normalizing the velocities by the value M and the gas pressure as

$$\Pi = \frac{p}{\rho_1 M^2}, \tag{16}$$

the system of shock conditions [2]–[5] reduces to

$$\tau + Hw^2 + \Pi = \tau^0 + \Pi^0 \tag{17}$$

and

$$\frac{1}{2}\tau^2 + E\tau w + \frac{1}{2}Fw^2 + \Pi = \frac{1}{2}(\tau^0)^2 + \Pi^0, \tag{18}$$

wherein

$$w = \tau\left(1 - \theta\frac{\tau^0 - 1}{\tau - 1}\right) \tag{19}$$

and

$$E = E(\tau) \equiv E(\varepsilon_2(\tau)), \quad \varepsilon_2(\tau) = (\tau - 1)/\tau, \tag{20}$$

$$H(\tau) = \frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}\tau} + \frac{E}{\tau}, \quad F(\tau) = \tau\frac{\mathrm{d}E}{\mathrm{d}\tau} - E. \tag{21}$$

For Maxwell's exertia,

$$E = \frac{\tau - 1}{2\tau}, \quad H = \frac{2\tau - 1}{4\tau^2}, \quad F = \frac{2 - \tau}{2\tau}. \tag{22}$$

The system of shock conditions [17] and [18] is sufficient to find any two of the following three values—a concentration (voidage) jump, a gas density jump and a shock propagation speed—when any third value of them is known, assuming that all hydrodynamic parameters in front of the shock are given. The velocities of the phases behind the shock can then be calculated in accordance with [13] and [14]. A jump of the gas pressure can be determined from the corresponding equation of state. When the hydrodynamic parameters on both sides of the shock are known, the fluid pressure p_1 both ahead and beyond the shock can be calculated from the relationship for the interphase pressure difference (Wallis 1989b, 1991):

$$p_1 - p = \frac{1}{2}\rho_1\varepsilon_1 w^2\frac{\mathrm{d}E}{\mathrm{d}\varepsilon_2}. \tag{23}$$

Eliminating the gas pressure from shock conditions [17] and [18], we obtain an equation for the relative velocity w in terms of τ (or the concentration of the dispersed phase ε_2) as follows:

$$\left(H - \frac{1}{2}F\right)w^2 - E\tau w + \tau - \tau^0 - \frac{1}{2}[\tau^2 - (\tau^0)^2] = 0. \tag{24}$$

As the relative velocity w is given by [19] as a function of τ and θ, [24] actually determines a relationship between the voidage and the gas density behind the shock. As soon as this relationsip is obtained, any one of the two shock conditions [17] or [18] gives an analog of the Hugoniot relationship connecting the jump of the gas density with the pressure jump and the shock propagation speed.

PROPAGATION OF A FINITE SMALL-AMPLITUDE SHOCK

Now we assume the shock to have a small but finite amplitude, so that the value of θ corresponding to the gas density beyond the shock can be represented in the form

$$\theta = 1 + \theta_1, \quad \theta_1 \ll 1. \qquad [25]$$

Ahead of the shock $\theta = \theta^0 = 1$. The jump of τ across the shock is now expanded in terms of θ_1 as follows:

$$[\tau] = \tau - \tau^0 = \gamma_1(\varepsilon_2^0)\theta_1 + \gamma_2(\varepsilon_2^0)\theta_1^2, \qquad [26]$$

where terms of third and higher orders in θ_1 are henceforth omitted.

We note that $\gamma_1(\varepsilon_2^0)$ corresponds to the linear analysis given in Sergeev & Wallis (1991), where the relationship between the voidage and density jump across the weak shock was obtained. From the results of the cited work, it immediately follows that

$$\gamma_1 = \frac{\varepsilon_2^0}{\varepsilon_1^0}\frac{E^0}{(\varepsilon_2^0)^2 + E^0}. \qquad [27]$$

Now the functions E, H and F in [24] can be expanded in terms of small $[\tau]$, using [22], and the values of w and w^2 behind the shock can be obtained from [19] as follows:

$$w = \frac{1}{(\varepsilon_2^0)^2}(\varepsilon_2^0 - (\varepsilon_1^0)^2[\tau])\left([\tau] - \frac{\varepsilon_2^0}{\varepsilon_1^0}\theta_1\right) + O(\theta_1^3) \qquad [28a]$$

and

$$w^2 = \frac{1}{(\varepsilon_2^0)^2}\left([\tau] - \frac{\varepsilon_2^0}{\varepsilon_1^0}\theta_1\right)^2 + O(\theta_1^3), \qquad [28b]$$

where τ is given by [26].

The second-order approximation of [24] follows from the expansions of E, H and F and [28a, b] in the form

$$\frac{E^0}{(\varepsilon_1^0)^2}\left(-\frac{1}{\gamma_1}[\tau] + \theta_1\right) + \left(-\frac{1}{2} + \frac{3\Gamma(\varepsilon_2^0)}{2(\varepsilon_2^0)^2}\right)[\tau]^2 - \frac{2\Gamma(\varepsilon_2^0)}{\varepsilon_1^0\varepsilon_2^0}\theta_1[\tau] + \frac{\Gamma(\varepsilon_2^0)}{2(\varepsilon_1^0)^2}\theta_1^2 = 0, \qquad [29]$$

where

$$\Gamma = -\frac{1}{2} - \frac{3(4\varepsilon_2 - 3)}{4\varepsilon_2^2}E - \frac{3\varepsilon_1}{2\varepsilon_2}\frac{dE}{d\varepsilon_2}, \qquad [30a]$$

$$\Lambda = \frac{8\varepsilon_2 - 7}{2\varepsilon_2^2}E + \frac{2\varepsilon_1}{\varepsilon_2}\frac{dE}{d\varepsilon_2} \qquad [30b]$$

and

$$\Omega = \frac{5 - 4\varepsilon_2}{4\varepsilon_2^2}E - \frac{\varepsilon_1}{2\varepsilon_2}\frac{dE}{d\varepsilon_2}. \qquad [30c]$$

The second-order expansion in θ_1 of the solution of [29] leads to the representation of $[\tau]$ in the form [26], where $\gamma_2(\varepsilon_2^0)$ is found to be of the form

$$\gamma_2 = \gamma_1\frac{(\varepsilon_1^0)^2}{E^0}\left\{\left[-\frac{1}{2} + \frac{3\Gamma^0}{2(\varepsilon_2^0)^2}\right]\gamma_1^2 - \frac{2\Gamma^0}{\varepsilon_1^0\varepsilon_2^0}\gamma_1 + \frac{\Gamma^0}{2(\varepsilon_1^0)^2}\right\} \qquad [31]$$

Taking into account [26], [28b] and [31] together with the expression of the function $H(\tau)$ in terms of $[\tau]$ we obtain from shock condition [17] the analog of the Hugoniot relationship for the finite small-amplitude shock as follows:

$$\theta_1\left\{\gamma_1+\theta_1\left[\gamma_2+\frac{(\varepsilon_2^0)^4(1+\varepsilon_2^0)}{\varepsilon_1^0((\varepsilon_2^0)^2+E^0)}\right]\right\}+\Pi-\Pi^0=0. \qquad [32]$$

Expanding the gas pressure in terms of a small density jump up to terms of second order,

$$\Pi-\Pi^0=\frac{1}{M^2}\left[c_0^2\frac{[\rho_2]}{\rho_1}+\left(\frac{\mathrm{d}^2p}{\mathrm{d}\rho_2^2}\right)_{\rho_2=\rho_2^0}\frac{[\rho_2]^2}{\rho_1^2}\right] \qquad [33]$$

where

$$[\rho_2]=\rho_2-\rho_2^0=-\rho_2^0\theta_1, \qquad [34]$$

and taking into account [15] and the relationship between the small voidage and gas concentration jumps (Sergeev & Wallis 1991),

$$[\rho_2]=-\rho_2^0\frac{(\varepsilon_2^0)^2+E^0}{\varepsilon_1^0\varepsilon_2^0E^0}[\varepsilon_2]+O(\theta_1^2) \qquad [35]$$

where

$$[\varepsilon_2]=\varepsilon_2-\varepsilon_2^0, \qquad [36]$$

we obtain from [32] the propagation speed of the finite small-amplitude shock wave in the form

$$D^2=D_0^2\left[1-\Phi(\varepsilon_2^0)[\varepsilon_2]-\frac{\rho_2^0}{\rho_1c_0^2}\frac{E^0+(\varepsilon_2^0)^2}{\varepsilon_1^0\varepsilon_2^0}\left(\frac{\mathrm{d}^2p}{\mathrm{d}\rho_2^2}\right)_{\rho_2=\rho_2^0}[\varepsilon_2]\right], \qquad [37]$$

where the terms of second and higher orders in $[\theta_1]$ are neglected ($\theta_1 \ll 1$ and $[\varepsilon_2] \ll 1$ for the small-amplitude shock). In [37], c_0 is the sound speed in the gas and D_0 is the propagation speed of an infinitely small concentration/density disturbance in an inertially-coupled two-phase dispersion, determined by Sergeev & Wallis (1991) as

$$D_0^2=c_0^2\frac{\rho_2^0}{\rho_1}\frac{(\varepsilon_2^0)+E^0}{\varepsilon_1^0\varepsilon_2^0E^0}. \qquad [38]$$

The function $\Phi(\varepsilon_2)$ in [37] is found to be of the form

$$\Phi=\frac{1}{\varepsilon_1^2\gamma_1^2}\left[\gamma_2+\frac{\varepsilon_2^4H}{\varepsilon_2^2(\varepsilon_2^2+E)}\right], \qquad [39]$$

where γ_2 is defined in [31] and H is determined by [9]. In the case of Maxwell's (1881) exertia, taking into account [8], [9] and [30], [39] reduces to

$$\Phi=\frac{5\varepsilon_2}{\varepsilon_1(1+2\varepsilon_2)}. \qquad [40]$$

For the case of Maxwell's exertia, the function $\Phi(\varepsilon_2)$ is represented in figure 1.

We note that, for small ε_2, [40] shows the asymptotic behavior of Φ as follows:

$$\Phi=5\varepsilon_2 \quad \text{at} \quad \varepsilon_2\to 0. \qquad [41]$$

Surprisingly, [41] gives almost exactly the values of Φ given by [40] within the interval $0 \leqslant \varepsilon_2 \leqslant 0.5$, with a maximum relative error of $< 10\%$ (it can hardly be expected that for a clean dispersion of bubbles in a fluid the voidage exceeds the value 0.5). At $\varepsilon_2 = 0$ and $\varepsilon_2 = 0.5$, [41] gives the exact values $\Phi = 0$ and $\Phi = 2.5$, respectively.

An asymptotic behavior of Φ at $\varepsilon_2 \to 1$ ($\varepsilon_1 \to 0$), although such values of ε_2 seem unrealistic, is as follows:

$$\Phi=\frac{5}{3\varepsilon_1} \quad \text{as} \quad \varepsilon_2\to 1, \quad \varepsilon_2\to 0. \qquad [42]$$

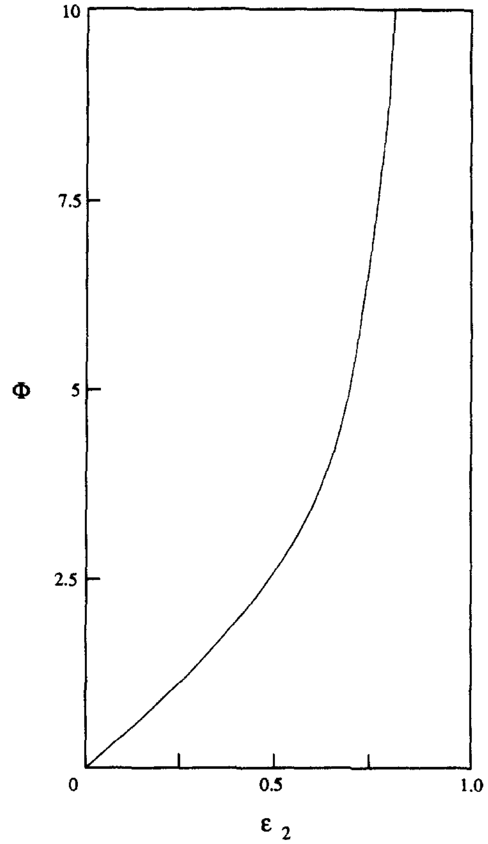

Figure 1. The function $\Phi(\varepsilon_2)$ for Maxwell's exertia.

Although it is more difficult to measure experimentally a gas density jump across the shock than a voidage jump, in some cases it can be useful to have a representation of the shock speed in terms of the jump of the gas density. From [37] and [35], we have

$$D^2 = D_0^2\left\{1 + \frac{[\rho_2]}{\rho_2^0}\left[\frac{\varepsilon_1^0\varepsilon_2^0 E^0}{(\varepsilon_2^0) + E^0}\,\Phi(\varepsilon_2^0) + \frac{1}{\rho_1 c_0^2}\left(\frac{\mathrm{d}^2 p}{\mathrm{d}\rho_2^2}\right)^0\right]\right\}. \qquad [43]$$

We note that the compression shock of the gas density propagates with a speed higher than the "speed of sound" in the two-phase media D_0 [the property $(\mathrm{d}^2 p/\mathrm{d}\rho_2^2)^0 > 0$ is, of course, assumed], while the rarefaction shock of the gas density propagates with a speed lower compared to D_0. It should be stressed that for a small-amplitude shock, a compression jump of the gas density is necessarily associated with a "rarefaction" jump of the concentration of the dispersed phase (and vice versa—a rarefaction jump of the gas density, if such a jump can be realized, should be associated with a "compression" jump of the concentration ε_2), as follows from [35]. The validity of the last statement for large-amplitude shocks is shown in the next section.

LARGE-AMPLITUDE SHOCKS

We start from the normalized system of shock conditions [17] and [18]. Here we shall consider the finite-amplitude shock only for the case of Maxwell's exertia, so that $E = \varepsilon_2/2$ and the functions H and F are of the form [9] and [10], respectively, or [22] in the case when the "gas-dynamics" variable $\tau = 1/\varepsilon_2$ is used.

For Maxwell's exertia, [24] for the relative velocity of the phases is of the form

$$\frac{\tau^2 - 1}{4\tau^2}\,w^2 - (\tau - 1)w + (\tau^0 - \tau)[(\tau^0 - 1) + (\tau - 1)] = 0. \qquad [44]$$

Since $1 \leqslant \tau \leqslant +\infty$, the condition for the solvability of this equation is of the form

$$A(\tau, \tau_0) = 2\tau^3 - 2\tau^2 - [(\tau^0)^2 - 2\tau^0 + 2]\tau - (\tau^0)^2 + 2\tau^0 \geqslant 0. \qquad [45]$$

It can be shown that the equation

$$A(\tau, \tau^0) = 0 \qquad [46]$$

has the only root in the interval $1 \leqslant \tau \leqslant +\infty$, so that condition [45] can be rewritten as

$$\tau \geqslant \tau_-(\varepsilon_2^0) \qquad [47]$$

or

$$\varepsilon_2 \geqslant \varepsilon_-(\varepsilon_2^0) = 1/(1 - \tau_-), \qquad [48]$$

where the functions $\tau_-(\varepsilon_2^0)$ and $\varepsilon_-(\varepsilon_2^0)$ are represented in figure 2.

The necessary condition for the realization of the shock [48] gives a lower limit to the concentration of the dispersed phase behind the front, which depends on the concentration ahead of the wave. Since $\varepsilon_- < \varepsilon_2^0$, [48] means that an inertial interphase interaction restricts an amplitude of the rarefaction shock of the concentration ε_2, such that

$$\|[\varepsilon_2]\| = \varepsilon_2^0 - \varepsilon_2 \leqslant \varepsilon_2^0 - \varepsilon_-(\varepsilon_2^0). \qquad [49]$$

Wave propagation in the form of a shock cannot be realized for concentrations $\varepsilon_2 < \varepsilon_-(\varepsilon_2^0)$ behind the wave front.

The solution of [44], taking into account [19] for the relative velocity of the phases, gives two roots for θ as a function of τ; one of them is negative for all values of $\tau \geqslant 1$ and, consequently, can be withdrawn from consideration, so that the dependence of the gas density in bubbles on the concentration of the dispersed phase behind the shock is given by the relationship

$$\theta = \frac{1}{(\tau^0 - 1)(\tau + 1)}[2\sqrt{(\tau - 1)A(\tau, \tau^0)} - (\tau - 1)^2], \qquad [50]$$

where the function $A(\tau, \tau^0)$ is given by [45]. The function $\theta(\tau)$ at $\varepsilon_2^0 = 0.1$, 0.25 and 0.4 is represented in figure 3.

An analog of the Hugoniot curve can now be obtained from [17], in which $\tau = \tau(\theta, \varepsilon_2^0)$ is the inverse function to [50]. We represent shock condition [17] in the form of the Hugoniot relationship as follows:

$$h(\theta) = \Pi^0 - \Pi(\theta), \qquad [51]$$

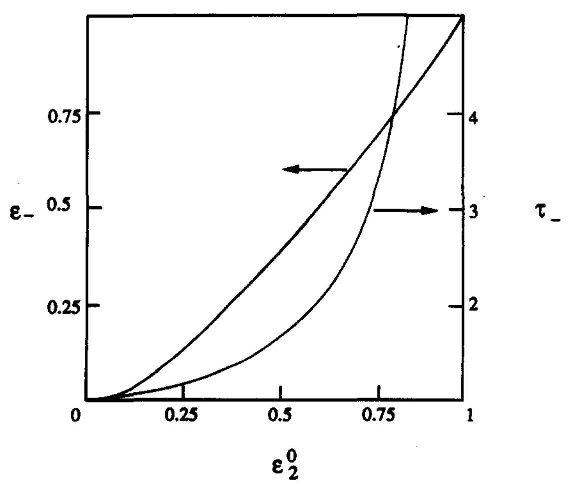

Figure 2. The functions $\tau_-(\epsilon_2^0)$ and $\epsilon_-(\epsilon_2^0)$.

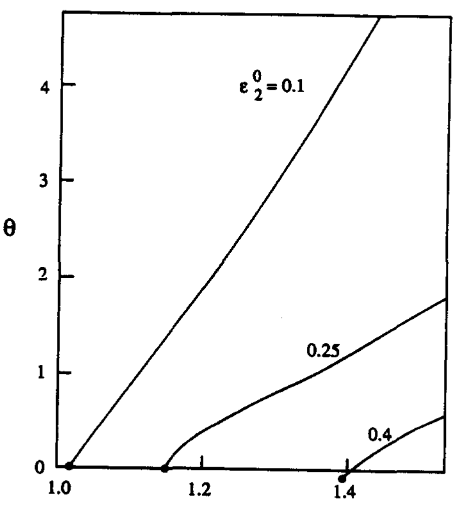

Figure 3. $\theta = \rho_2^0/\rho_2$ as a function of $\tau = 1/(1 - \epsilon_2)$; the ● points correspond to the minimum values of $\tau = \tau_-(\epsilon_2^0)$.

where, taking into account [19] and the form of $H(\tau)$ in [22], the function $h(\theta)$ is of the form

$$h(\theta) = \tau(\theta) - \tau(\varepsilon_2^0) + \frac{2\tau(\theta) - 1}{4}\left[1 - \theta\frac{\tau(\varepsilon_2^0) - 1}{\tau(\theta) - 1}\right]^2. \qquad [52]$$

Since the function $\theta(\tau)$ is monotonic within the interval $\tau_-(\varepsilon_2^0) \leqslant \tau \leqslant +\infty$, the functions $\tau(\theta)$ and $h(\theta)$ are defined within the interval $\theta_-(\varepsilon_2^0) \leqslant \theta < +\infty$. Here θ_- corresponds to the value τ_- (or ε_-). As $A(\tau_-, \tau_0) = 0$, from [50] it follows that

$$\theta_- = -\frac{(\tau_- - 1)^2}{(\tau^0 - 1)(\tau_- + 1)}, \qquad [53]$$

where $\tau_-(\varepsilon_2^0)$ is given in figure 2.

From [53] it follows that $\theta_- < 0$ for all values of ε_2^0. Since $\theta = \rho_2^0/\rho_2 > 0$, the functions $\tau(\theta)$ and $h(\theta)$ can be considered as defined within the interval

$$0 < \theta < +\infty. \qquad [54]$$

The function $h(\theta)$ is represented in figure 4 for $\varepsilon_2^0 = 0.1$, 0.25 and 0.4. Below we name the curves $h = h(\theta)$ and $h = \Pi^0 - \Pi(\theta)$ as the h-curve and Π-curve, respectively. The h- and Π-curves intersect at the point $\theta = 1$, $h = 0$, which corresponds to the undisturbed two-phase flow ahead of the wave front. For any type of gas behavior inside bubbles the Π-curve increases with θ and has the form quantitatively represented in figure 5(a–c) by the dashed lines with the horizontal asymptote $h = \Pi^0$.

An important result follows from the fact that the function $\theta(\tau)$ increases with τ. Since $\tau = 1/(1 - \varepsilon_2)$ increases with ε_2, the rarefaction shock of the concentration of the dispersed phase is necessarily associated with the compression shock of the gas density inside bubbles ($\rho_2 > \rho_2^0$ as $\varepsilon_2 < \varepsilon_2^0$) and the compression concentration shock is associated with the rarefaction density shock ($\rho_2 < \rho_2^0$ as $\varepsilon_2 > \varepsilon_2^0$). For small-amplitude shocks this result has been obtained by Sergeev & Wallis (1991) and was mentioned above (see [35]).

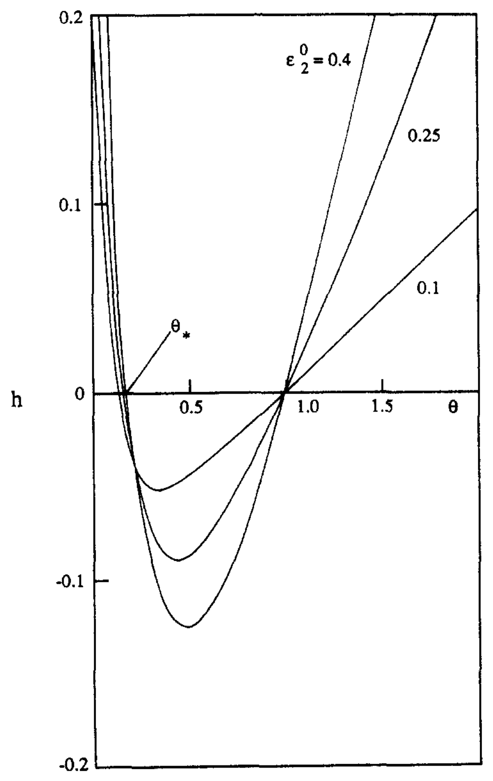

Figure 4. The function $h(\theta)$.

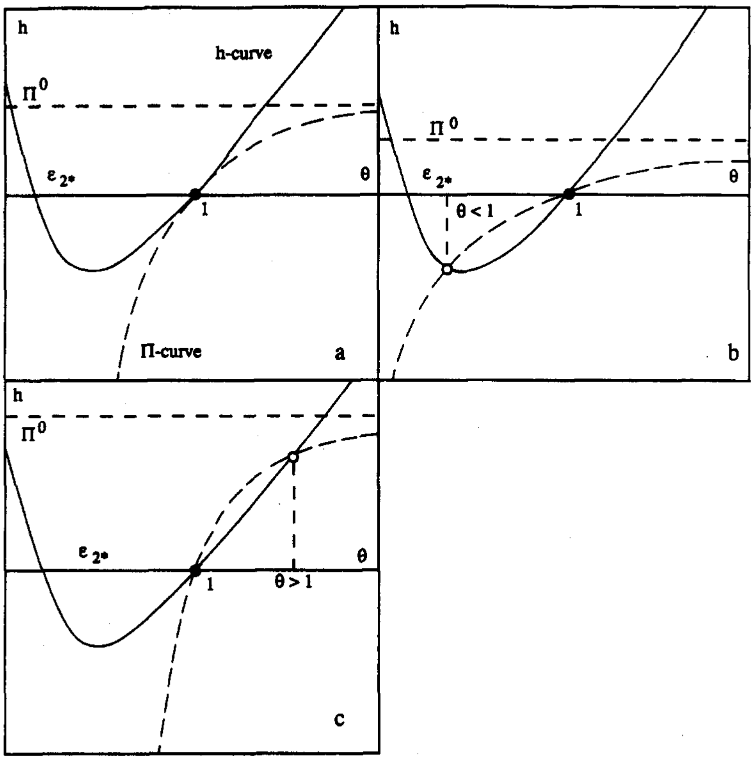

Figure 5. The sketch of mutual intersections of the h-curve (———) with the Π-curve (- - -): (a) linear disturbance, the curves are tangential; (b) compression shock of the gas density in bubbles associated with the rarefaction shock of the concentration of the dispersed phase; (c) rarefaction shock of the gas density associated with the compression shock of the concentration.

Obviously, there are three possibilities for the mutual intersection of the h- and Π-curves shown in figure 5(a–c):

1. The curves are tangential at $\theta = 1$, $h = 0$, so that $dh/d\theta = d(\Pi^0 - \Pi)/d\theta$ at $\theta = 1$. Using the properties of the functions $h = \Pi^0 - \Pi(\theta)$ and $h = h(\theta)$, it can be proved that there are no other points of intersection. This case corresponds to the propagation of small (linear) concentration/density disturbances near the steady state $\theta = 1$ ($\rho_2 = \rho_2^0$), $\varepsilon_2 = \varepsilon_2^0$ when the propagation speed $D = D_0$ is given by [38].
2. Besides the point $\theta = 1$, $h = 0$, the h-curve intersects the "negative" part of the Π-curve at $h < 0$, $\theta < 1$ (an absence of other intersections can be proved for this case as well), so that

$$\frac{dh}{d\theta} > \frac{d(\Pi^0 - \Pi)}{d\theta} \quad \text{at} \quad \theta = 1. \qquad [55]$$

The case under consideration corresponds to the compression jump of the gas density inside bubbles across the wave front and to the associated rarefaction concentration jump.

We now consider the behavior of the h-curve in the interval $0 < \theta < 1$, which corresponds to the compression jump of the gas density. Besides the point $\theta = 1$, $h = 0$, the h-curve intersects the θ-axis at $\theta = \theta_* < 1$, where θ_* and the corresponding value of the concentration $\varepsilon_2{}^*$ can be obtained from the system of the algebraic equation [52] at $h = 0$ and [50]. The function $\theta_*(\varepsilon_2^0)$ is represented in figure 6. Since $\Pi^0 - \Pi < 0$ for the compression shock of the gas density [see figure 5(b)], $\theta_*(\varepsilon_2^0)$ gives the lower limit to the value $\theta = \rho_2^0/\rho_2$ behind the shock. It means that the gas density in the dispersed phase behind the wave front is restricted, such that

$$\rho_2 \leqslant \rho_2^0 \theta_*^{-1}(\varepsilon_2^0). \qquad [56]$$

Inequality [56] gives the maximum compression of the gas across the shock ρ_2/ρ_2^0 as θ_*^{-1}, which is a function of ε_2^0. Since θ_* increases with ε_2^0, the maximum compression decreases with the concentration of the dispersed phase in the undisturbed dispersion. As $\varepsilon_2^0 \to 1$, $\theta_* \to 0.217$, hence the lowest maximum compression $[(\rho_2/\rho_2^0)_*]_{\min} = 4.61$ corresponds to high concentrations of the dispersed phase.

Inequality [56] shows that the strength of the gas density jump is limited as follows:

$$[\rho_2] \leqslant [\rho_2]_* = \rho_2^0\left(\frac{1}{\theta_*(\varepsilon_2^0)} - 1\right). \qquad [57]$$

The maximum value of the strength of the density compression jump $[\rho_2]_*$ as a function of ε_2^0 is represented in figure 7.

As shown above, the concentration jump of the gas density is associated with the rarefaction jump of the concentration of the dispersed phase. The direct numerical solution of system [52] at $h = 0$ and [50] shows that the concentration $\varepsilon_2{}^*$ corresponding to the value θ_* is such that $\varepsilon_{2*} > \varepsilon_{2-}$, so that ε_{2*} (in figure 8) gives the lower limit to the concentration behind the wave front (the relationship between the gas density and the concentration beyond the shock is given by [50] in which $\tau = 1/(1-\varepsilon_2)$ and $\theta = \rho_2^0/\rho_2$).

We can now conclude that the obtained restrictions of the shock strength are only due to hydrodynamic effects, so that the shocks with $\varepsilon_2 < \varepsilon_{2*}$ and $\theta < \theta_*(\rho_2 > \rho_2^0\theta_*^{-1})$ are "inertially" forbidden. The detailed analysis of the thermodynamic processes in gas bubbles can add some other restrictions as well.

The propagation speed of the shock D can now be found from [51] in dependence on the given value of the gas density ρ_2 or the concentration ε_2 behind the front (or through the values of the jumps $[\rho_2]$ or $[\varepsilon_2]$ of these parameters across the shock). The propagation speed D is determined by the value of the parameter M, [15], in the expression for the function Π, [16]. Taking into account that for

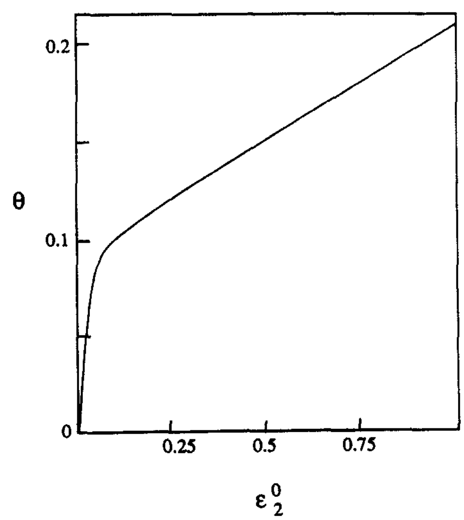

Figure 6. The function $\theta_*(\epsilon_2^0)$.

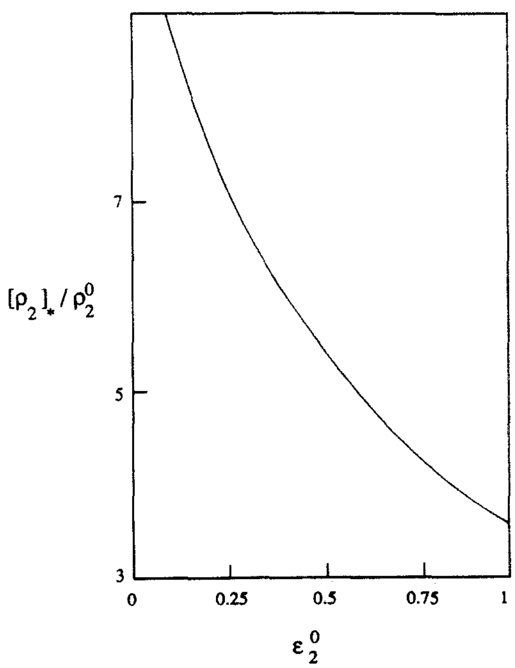

Figure 7. The maximum value of the strength of the density compression jump as a function of ϵ_2^0.

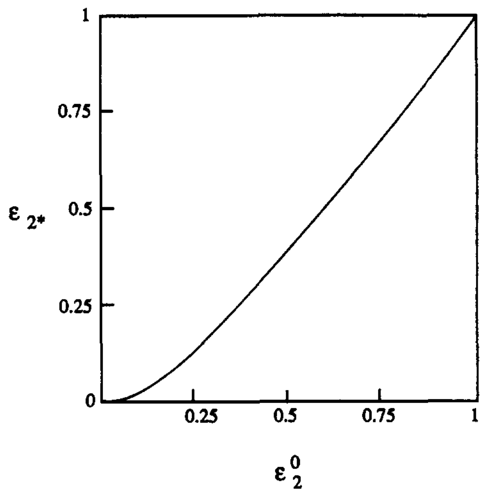

Figure 8. The lower limit of the concentration of the dispersed phase behind the shock vs the concentration ahead of the wave (in steady state).

the undisturbed dispersions $M = M_0 = -\varepsilon_1^0 D_0$, where D_0 is the speed of "sound", [38], the h- and Π-curves are tangential, so that $\mathrm{d}h/\mathrm{d}\theta = \mathrm{d}(\Pi^0 - \Pi)/\mathrm{d}\theta$ at $\theta = 1$; we immediately find from [51], [56], [16] and [15] that in the case of the compression jump of the gas density in the dispersed phase

$$D > D_0. \qquad [58]$$

This result means that the concentration/density shock wave associated with the compression jump of the gas density inside bubbles across the wave front propagates with a higher speed than the speed of "sound" in the two-phase dispersion.

3. The second point of the intersection belongs to the "positive" part of the Π-curve at $h > 0$, $\theta > 1$. Here

$$\frac{\mathrm{d}h}{\mathrm{d}\theta} < \frac{\mathrm{d}(\Pi^0 - \Pi)}{\mathrm{d}\theta} \quad \text{at} \quad \theta = 1. \qquad [59]$$

This situation corresponds to a rarefaction jump of the gas density inside bubbles across the shock. From [58], [16] and [15] it follows that a rarefaction shock of the gas density associated with the compression of the concentration of the dispersed phase propagates with a speed lower than the speed of "sound", so that

$$D < D_0, \qquad [60]$$

where D_0 is given by [38].

An existence of the rarefaction shock of the gas density in bubbles is, of course, questionable. Nevertheless, because we are not going to discuss the thermodynamic aspects of the behavior of the two-phase dispersion in this paper [such a consideration should necessarily include a detailed model of the interphase heat transfer, see Nigmatulin (1978)], the discussion on the important problem of a possibility of a realization of the rarefaction jump of the gas density inside bubbles is left for further development of the model under consideration.

It should be underlined here that, while the strength of a compression shock of the gas density is limited by the value $[\rho_2]_*$ given by [57], the strength of a rarefaction shock of the gas density is unlimited within the frame of the hydrodynamic theory under consideration. Certain limitations to rarefaction shocks of the gas density inside bubbles should appear with a detailed accounting of thermodynamic and heat transfer properties of the dispersion.

In conclusion, we find a relationship between the strength and the propagation speed of the shock. From [51], taking into account [16] and [15], we have

$$D^2 = \frac{p^0 - p}{\rho_1 (\varepsilon_1^0)^2 h(\theta)}. \qquad [61]$$

Taking into account that the propagation speed V of the shock wave with the same density jump in a pure gas can be written as (Courant & Friedrichs 1948)

$$V = \left(\frac{\rho_2}{\rho_2^0} \frac{p - p^0}{\rho^2 - \rho_2^0}\right)^{1/2}, \qquad [62]$$

we represent the propagation speed of the shock in the two-phase dispersion in the form

$$D = \left(\frac{\rho_2^0}{\rho_1}\right)^{1/2} V\bar{D}, \qquad [63]$$

where

$$\bar{D} = \frac{1}{\varepsilon_1^0}\left[\frac{\theta - 1}{h(\theta)}\right]^{1/2}. \qquad [64]$$

Such a form of representation is very convenient for further analysis, because $\bar{D}$ does not depend on the type of thermodynamic process inside bubbles or on interphase heat transfer.

The normalized speed of the shock $\bar{D}$ as a function of $\theta = \rho_2^0/\rho_2$ is given in figure 9 for the concentration of the dispersed phase ahead of the wave front $\varepsilon_2^0 = 0.1$, 0.25 and 0.4. Figure 10 illustrates the behavior of $\bar{D}$ as a function of ε_2 for the same values of ε_2^0. The • points in both figures correspond to the propagation velocity of linear (small-amplitude) waves in the two-phase dispersion at $\theta = 1$, $\varepsilon_2 = \varepsilon_2^0$. Direct comparison shows that the values of $\bar{D}$ for the parameters corresponding to the • points in figures 9 and 10 coincide with $\sqrt{\rho_1/\rho_2^0} D_0 c_0$, where D_0 is the propagation speed of infinitely small concentration/density disturbances calculated from [38]. The vertical asymptotes in figures 9 and 10 correspond to $\theta = \theta_*(\varepsilon_2^0)$ and $\varepsilon_2 = \varepsilon_{2*}(\varepsilon_2^0)$, respectively. The extreme character of the function $D(\varepsilon_2)$ [or $D(\theta)$] with the slow increase at

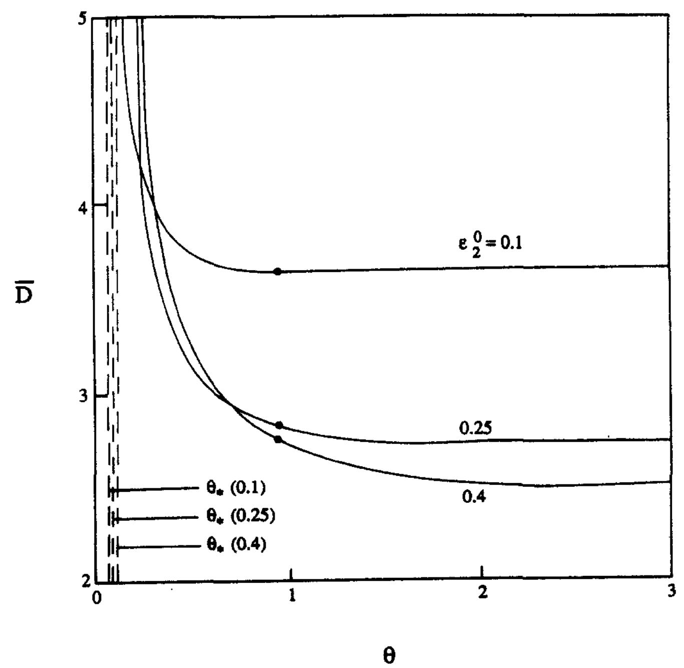

Figure 9. The normalized speed of the concentration/density shock wave $\bar{D} = \sqrt{\rho_1/\rho_2^0} D/V$, where V is the speed of the shock wave in a pure gas at the same density jump vs $\theta = \rho_2^0/\rho_2$ behind the wave front; the • points correspond to the speed of "sound" in the two-phase dispersion in the steady state ($\theta = 1$, $\epsilon_2 = \epsilon_2^0$); vertical asymptotes correspond to $\theta_*(\epsilon_2^0)$.

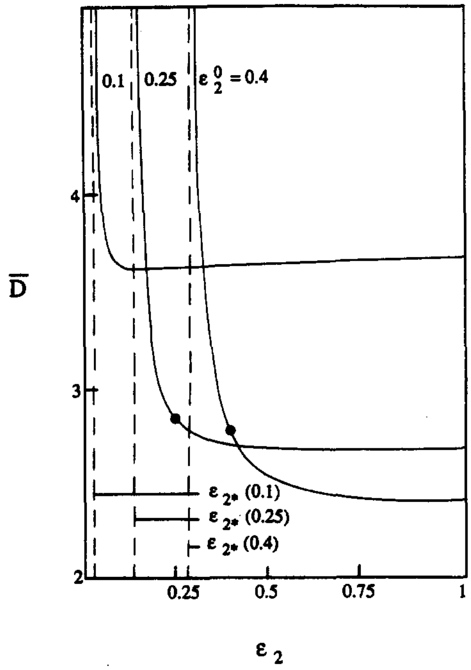

Figure 10. $\bar{D}$ as a function of the concentration of the dispersed phase behind the wave front; the • points are described in the caption to figure 9; vertical asymptotes correspond to $\varepsilon_{2*}(\varepsilon_2^0)$.

high ε_2 or θ, clearly seen at $\varepsilon_2^0 = 0.1$ in figure 10, does not have any physical interpretation. Indeed, the physical (not normalized) speed of the shock D in [63] monotonically decreases with θ (or ε_2) because of the rapid decrease of the speed of the shock wave in the pure gas V with θ (in spite of the slow increase of $\bar{D}$ at high θ).

Example

We consider an air–water two-phase dispersion at normal conditions with the parameters $\varepsilon_2^0 = 0.25$ ($\varepsilon_1^0 = 0.75$) in the steady state. We suppose an isothermal gas process within the bubbles, so that the speed of sound in the pure gas $c_0 = 330$ m/s. From [38] we find the speed of "sound" in the two-phase dispersion $D_0 = 33.6$ m/s. We consider the concentration/density shock wave with the concentration of the dispersed phase behind the front $\varepsilon_2 = 0.18$ (the concentration jump $[\varepsilon_2] = \varepsilon_2 - \varepsilon_2^0 = -0.07$). From [50], [45] and [11] we find $\theta = 0.485$ behind the shock, so that $\rho_2 = 2.666$ kg/m^3 and the density jump $[\rho_2] = 1.373$ kg/m^3. From [64], for the shock wave with the same density jump in a pure air at an isothermal process, we obtain $V = 351$ m/s. The value of $\bar{D}$ for the calculated value of θ is found from [66] as $\bar{D} = 3.235$, so that the propagation speed of the shock $D = 40.83$ m/s. So, for the given concentration jump $[\varepsilon_2] = -0.07$, the type of thermodynamic process in the gas and the values of the parameters of the steady state (i.e. ahead of the wave; in particular, $\varepsilon_2^0 = 0.25$), the propagation speed of the shock is 8.08 times less than the speed of the sound in the pure gas, 8.6 times less than the propagation speed of the shock with the same density jump in the pure gas and 22% higher than the speed of "sound" in the two-phase dispersion.

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